Fuzzy Set Theory

Chapter 6

Problems in the real world quite often turn out to be complex owing to an element of uncertainty either in the parameters which define the problem or in the situations in which the problem occurs.

Although probability theory has been an age old and effective tool to handle uncertainty, it can be applied only to situations whose characteristics are based on random processes, that is, processes in which the occurrence of events is strictly determined by chance. However, in reality, there turn out to be problems, a large class of them whose uncertainty is characterized by a nonrandom process. Here, the uncertainty may arise due to partial information about the problem, or due to information which is not fully reliable, or due to inherent imprecision in the language with which the problem is defined, or due to receipt of information from more than one source about the problem which is conflicting.

It is in such situations that *fuzzy set theory* exhibits immense potential for effective solving of the uncertainty in the problem. *Fuzziness* means 'vagueness'. Fuzzy set theory is an excellent mathematical tool to handle the uncertainty arising due to vagueness. Understanding human speech and recognizing handwritten characters are some common instances where fuzziness manifests.

It was Lotfi A. Zadeh who propounded the fuzzy set theory in his seminal paper (Zadeh, 1965). Since then, a lot of theoretical developments have taken place in this field. It is however, the Japanese who seem to have fully exploited the potential of fuzzy sets by commercializing the technology. More than 2000 patents have been acquired by the Japanese in the application of the technique and the area spans a wide spectrum, from consumer products and electronic instruments to automobile and traffic monitoring systems.

6.1 FUZZY VERSUS CRISP

Consider the query, "Is water colourless?" The answer to this is a definite *Yes/True*, or *No/False*, as warranted by the situation. If "yes"/"true" is accorded a value of 1 and "no"/"false" is accorded a value of 0, this statement results in a 0/1 type of situation. Such a logic which demands a binary (0/1) type of handling is termed *crisp* in the domain of fuzzy set theory. Thus, statements such as "Temperature is 32°C", "The running time of the program is 4 seconds" are examples of crisp situations.

On the other hand, consider the statement, "Is Ram honest?" The answer to this query need not be a definite "yes" or "no". Considering the degree to which one knows Ram, a variety of answers spanning a range, such as "extremely honest", "extremely dishonest", "honest at times", "very honest" could be generated. If, for instance, "extremely honest" were to be accorded a value of 1, at the high end of the spectrum of values, "extremely dishonest" a value of 0 at the low end of the spectrum, then, "honest at times" and "very honest" could be assigned values of 0.4 and 0.85 respectively. The situation is therefore so fluid that it can accept values between 0 and 1, in contrast to the earlier one which was either a 0 or 1. Such a situation is termed *fuzzy*. Figure 6.1 shows a simple diagram to illustrate fuzzy and crisp situations.

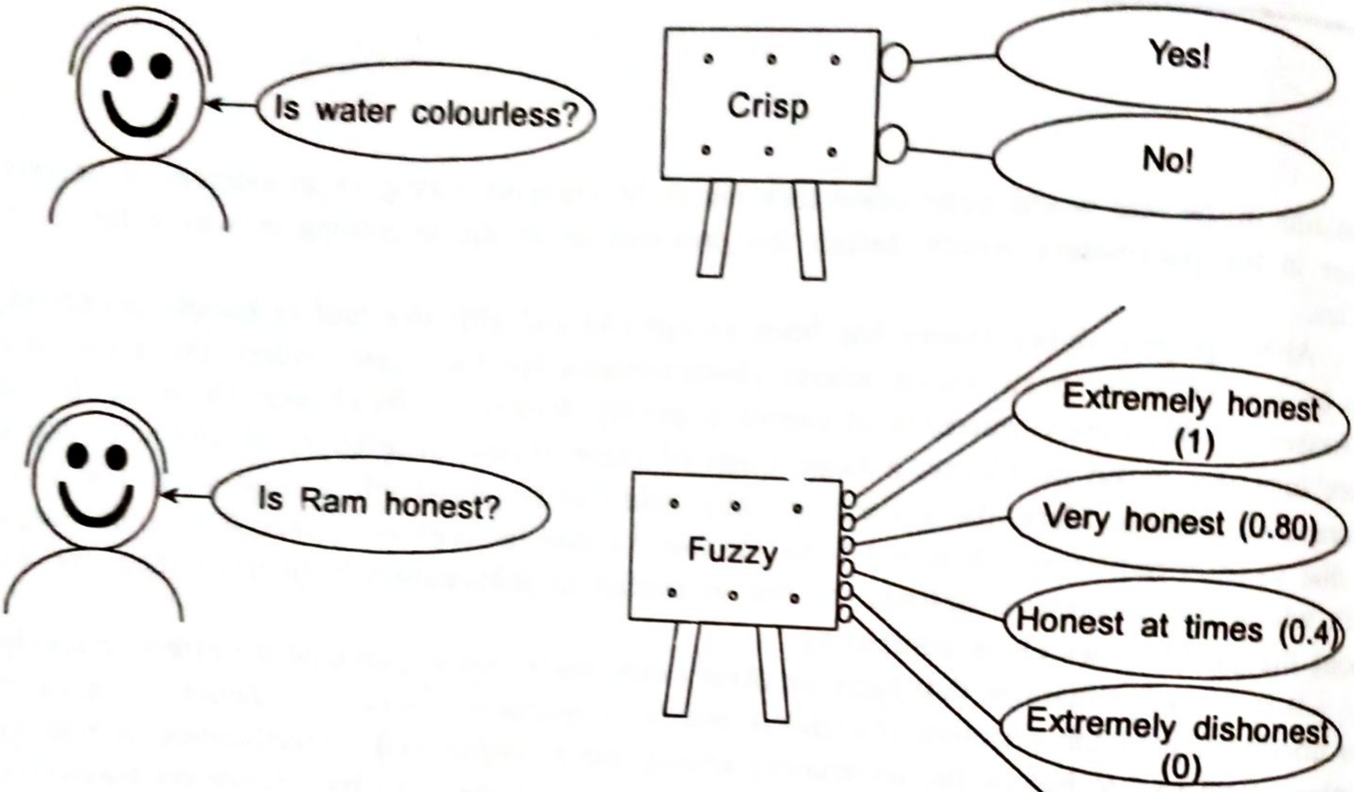


Fig. 6.1 Fuzzy versus crisp.

Classical set theory also termed *crisp set theory* and propounded by George Cantor is fundamental to the study of fuzzy sets. Just as Boolean logic had its roots in the theory of crisp sets, fuzzy logic has its roots in the theory of fuzzy sets (refer Fig. 6.1).

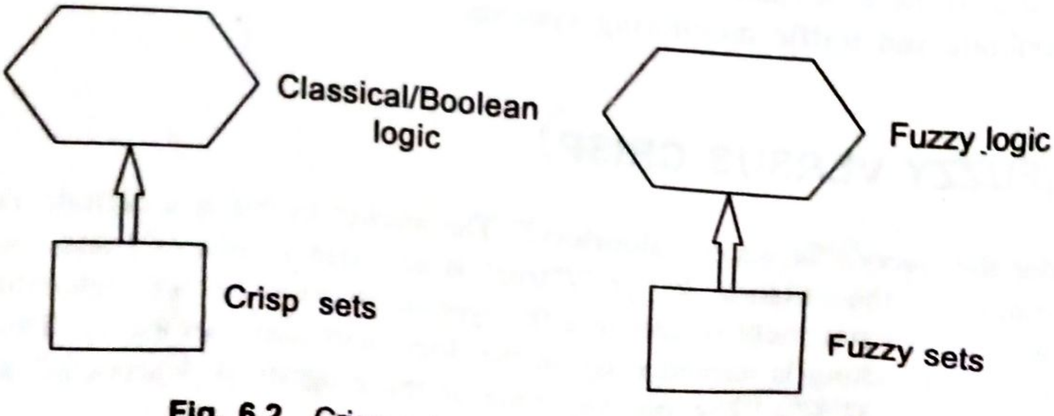


Fig. 6.2 Crisp sets and fuzzy sets.

We now briefly review crisp sets and its operations before a discussion on fuzzy sets is undertaken.

6.2 CRISP SETS

Universe of discourse

The *universe of discourse* or *universal set* is the set which, with reference to a particular context, contains all possible elements having the same characteristics and from which sets can be formed. The universal set is denoted by E.

Example

(i) The universal set of all numbers in Euclidean space.

(ii) The universal set of all students in a university.

Set

A set is a *well defined* collection of objects. Here, well defined means the object either belongs to or does not belong to the set (observe the "crispness" in the definition).

A set in certain contexts may be associated with its universal set from which it is derived. Given a set A whose objects are $a_1, a_2, a_3, \ldots, a_n$, we write A as $A = \{a_1, a_2, \ldots, a_n\}$. Here, $a_1, a_2, \ldots, a_n$ are called the *members* of the set. Such a form of representing a set is known as *list form*.

Example

A = {Gandhi, Bose, Nehru}

B = {Swan, Peacock, Dove}

A set may also be defined based on the properties the members have to satisfy. In such a case, a set A is defined as

$$A = \{x \mid P(x)\} \tag{6.1}$$

Here, $P(x)$ stands for the property P to be satisfied by the member x. This is read as 'A is the set of all X such that $P(x)$ is satisfied'.

Example

$A = \{x \mid x \text{ is an odd number}\}$

$B = \{y \mid y > 0 \text{ and } y \bmod 5 = 0\}$

Venn diagram

Venn diagrams are pictorial representations to denote a set. Given a set A defined over a universal set E, the Venn diagram for A and E is as shown in Fig. 6.3.

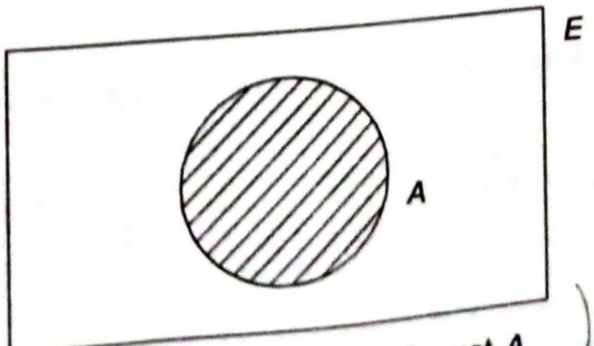


Fig. 6.3 Venn diagram of a set A.

Example

In Fig. 6.3, if E represents the set of university students then A may represent the set of female students.

Membership

An element x is said to be a member of a set A if x belongs to the set A. The membership is indicated by '$\in$' and is pronounced "belongs to". Thus, $x \in A$ means x belongs to A and $x \notin A$ means *x does not belong to A*.

Example

Given $A = \{4, 5, 6, 7, 8, 10\}$, for $x = 3$ and $y = 4$, we have $x \notin A$ and $y \in A$

Here, observe that each element either belongs to or does not belong to a set. The concept of membership is definite and therefore crisp (1—belongs to, 0—does not belong to). In contrast, as we shall see later, a fuzzy set accommodates membership values which are not only 0 or 1 but *anything* between 0 and 1.

Cardinality

The number of elements in a set is called its cardinality. Cardinality of a set A is denoted as $n(A)$ or $|A|$ or $\#A$.

Example

If $A = \{4, 5, 6, 7\}$ then $|A| = 4$

Family of sets

A set whose members are sets themselves, is referred to as a family of sets.

Example

$A = \{\{1, 3, 5\}, \{2, 4, 6\}, \{5, 10\}\}$ is a set whose members are the sets $\{1, 3, 5\}$, $\{2, 4, 6\}$, and $\{5, 10\}$.

Null Set/Empty Set

A set is said to be a *null set* or *empty set* if it has no members. A null set is indicated as $\varnothing$ or $\{\}$ and indicates an impossible event. Also, $|\varnothing| = 0$.

Example

The set of all prime ministers who are below 15 years of age.

Singleton Set

A set with a single element is called a *singleton set*. A singleton set has cardinality of 1.

Example

If $A = \{a\}$, then $|A| = 1$

Subset

Given sets A and B defined over E the universal set, A is said to be a *subset* of B if A is fully contained in B, that is, every element of A is in B.

Denoted as $A \subset B$, we say that A is a subset of B, or A is a *proper subset* of B. On the other hand, if A is contained in or equivalent to that of B then we denote the subset relation as $A \subseteq B$. In such a case, A is called the *improper* subset of B.

Superset

Given sets A and B on E the universal set, A is said to be a *superset* of B if every element of B is contained in A.

Denoted as $A \supset B$, we say A is a superset of B or A contains B. If A contains B and is equivalent to B, then we denote it as $A \supseteq B$.

Example

Let $A = \{3, 4\}$ $B = \{3, 4, 5\}$ and $C = \{4, 5, 3\}$

Here, $A \subset B$, and $B \supset A$
$C \subseteq B$, and $B \supseteq C$

Power set

A *power set* of a set A is the set of all possible subsets that are derivable from A including null set.

A power set is indicated as $P(A)$ and has cardinality of $|P(A)| = 2^{|A|}$

Example

Let $A = \{3, 4, 6, 7\}$

$P(A) = \{\{3\}, \{4\}, \{6\}, \{7\}, \{3, 4\}, \{4, 6\}, \{6, 7\}, \{3, 7\}, \{3, 6\}, \{4, 7\},$
$\{3, 4, 6\}, \{4, 6, 7\}, \{3, 6, 7\}, \{3, 4, 7\}, \{3, 4, 6, 7\}, \varnothing\}$

Here, $|A| = 4$ and $|P(A)| = 2^4 = 16$.

6.2.1 Operations on Crisp Sets

Union ($\cup$)

The union of two sets A and B ($A \cup B$) is the set of all elements that belong to A or B or both.

$$A \cup B = \{x / x \in A \text{ or } x \in B\} \tag{6.2}$$

Example

Given $A = \{a, b, c, 1, 2\}$ and $B = \{1, 2, 3, a, c\}$, we get $A \cup B = \{a, b, c, 1, 2, 3\}$
Figure 6.4 illustrates the Venn diagram representation for $A \cup B$

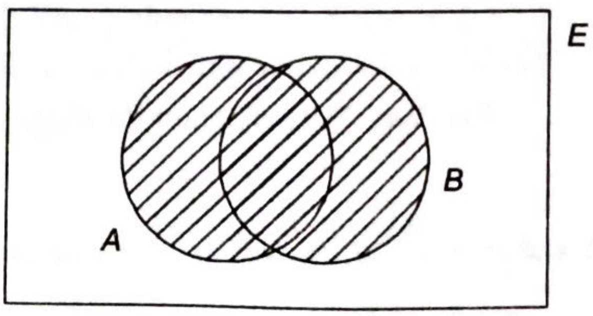


Fig. 6.4 Venn diagram for $A \cup B$.

Intersection ($\cap$)

The intersection of two sets A and B ($A \cap B$) is the set of all elements that belong to A and B

$$A \cap B = \{x / x \in A \text{ and } x \in B\}$$

Any two sets which have $A \cap B = \varnothing$ are called *Disjoint Sets*.

Example

Given $A = \{a, b, c, 1, 2\}$ and $B = \{1, 2, 3, a, c\}$, we get $A \cap B = \{a, c, 1, 2\}$

Figure 6.5 illustrates the Venn diagram for $A \cap B$

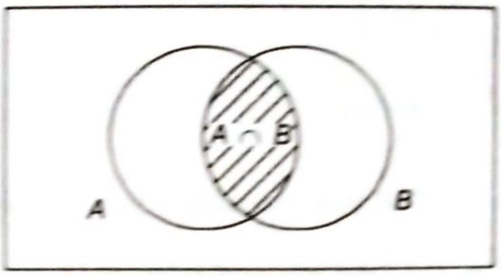


Fig. 6.5 Venn diagram for $A \cap B$.

Complement (^{c})

The complement of a set A ($\bar{A} \mid A^c$) is the set of all elements which are in E but not in A.

$$A^c = \{x / x \notin A,\ x \in E\} \quad (6.4)$$

Example

Given $X = \{1, 2, 3, 4, 5, 6, 7\}$ and $A = \{5, 4, 3\}$, we get $A^c = \{1, 2, 6, 7\}$

Figure 6.6 illustrates the Venn diagram for A^c.

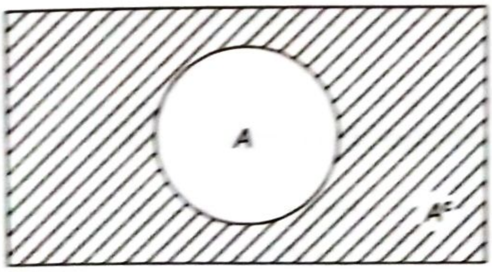


Fig. 6.6 Venn diagram for A^c.

Difference (−)

The difference of the set A and B is $A - B$, the set of all elements which are in A but not in B

$$A - B = \{x \mid x \in A \text{ and } x \notin B\} \quad (6.5)$$

Example

Given $A = \{a, b, c, d, e\}$ and $B = \{b, d\}$, we get $A - B = \{a, c, e\}$

Figure 6.7 illustrates the Venn diagram for $A - B$.

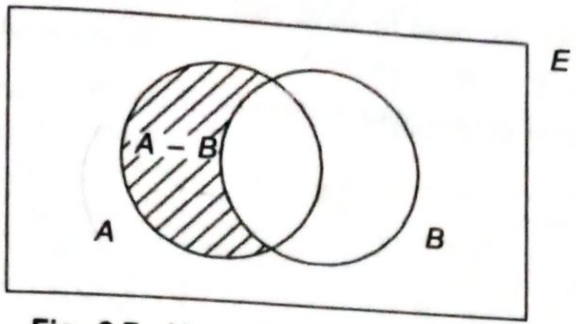


Fig. 6.7 Venn diagram for $A - B$.

6.2.2 Properties of Crisp Sets

The following properties of sets are important for further manipulation of sets.

Commutativity:
$$A \cup B = B \cup A$$
$$A \cap B = B \cap A \quad (6.6)$$

Associativity:
$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C) \quad (6.7)$$

Distributivity:
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \quad (6.8)$$

Idempotence:
$$A \cup A = A$$
$$A \cap A = A \quad (6.9)$$

Identity:
$$A \cup \varnothing = A$$
$$A \cap E = A$$
$$A \cap \varnothing = \varnothing$$
$$A \cup E = E \quad (6.10)$$

Law of Absorption:
$$A \cup (A \cap B) = A$$
$$A \cap (A \cup B) = A \quad (6.11)$$

Transitivity: If $A \subseteq B$, $B \subseteq C$ then $A \subseteq C$ (6.12)

Involution:
$$(A^c)^c = A \quad (6.13)$$

Law of the Excluded Middle:
$$A \cup A^c = E \quad (6.14)$$

Law of Contradiction:
$$A \cap A^c = \varnothing \quad (6.15)$$

De Morgan's laws:
$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c \quad (6.16)$$

All the properties could be verified by means of Venn diagrams.

Example 6.1

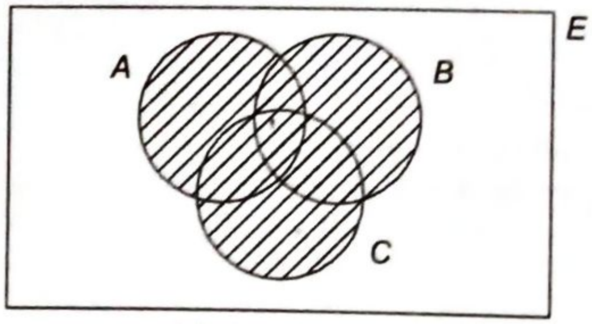
Given three sets A, B, and C. Prove De Morgan's laws using Venn diagrams.

Solution

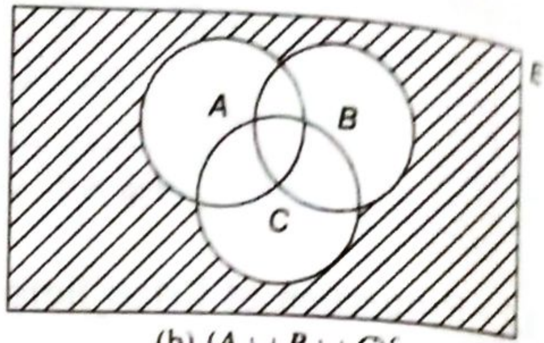
To prove De Morgan's laws, we need to show that

(i) $(A \cup B \cup C)^c = A^c \cap B^c \cap C^c$

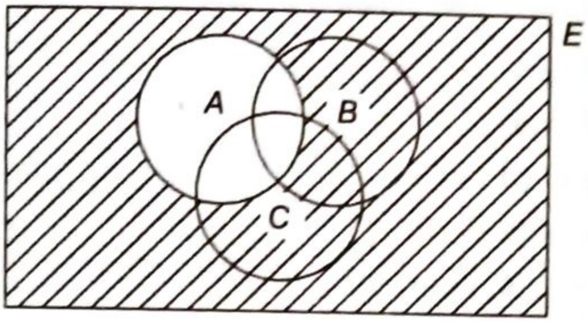
(ii) $(A \cap B \cap C)^c = A^c \cup B^c \cup C^c$



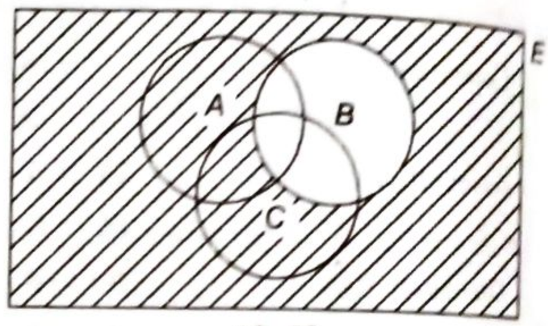
(a) $A \cup B \cup C$



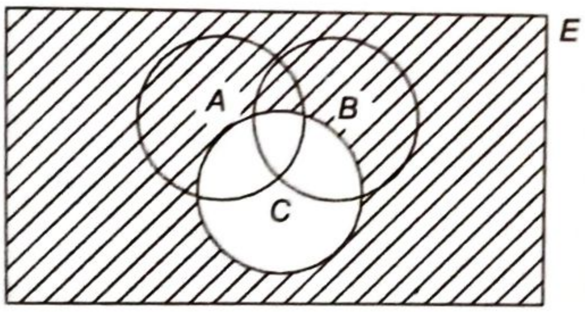
(b) $(A \cup B \cup C)^c$



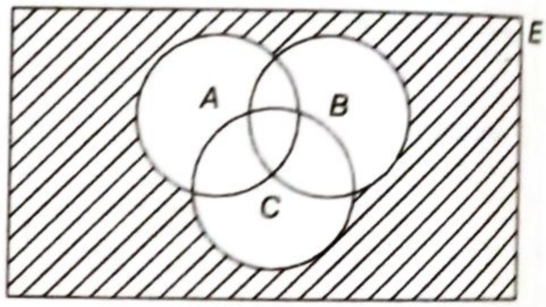
(c) A^c



(d) B^c

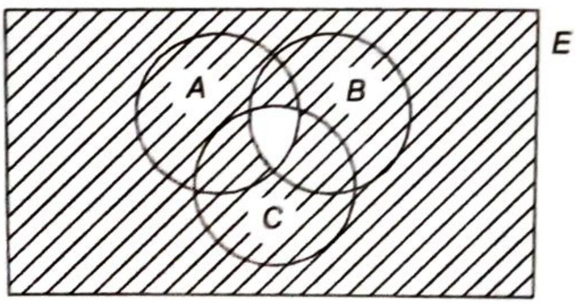


(e) C^c

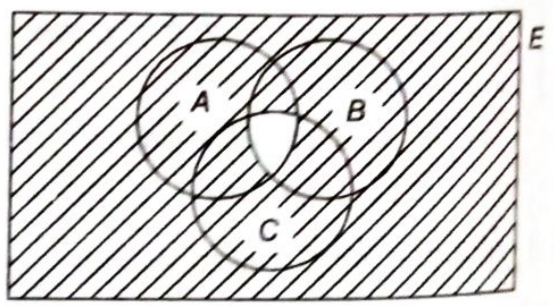


(f) $A^c \cap B^c \cap C^c$

(i) Here, it can be seen that $(A \cup B \cup C)^c = A^c \cap B^c \cap C^c$.



(g) $(A \cap B \cap C)^c$



(h) $(A^c \cup B^c \cup C^c)$

(ii) Figures (g) to (h) show that $(A \cap B \cap C)^c = A^c \cup B^c \cup C^c$.

Example 6.2

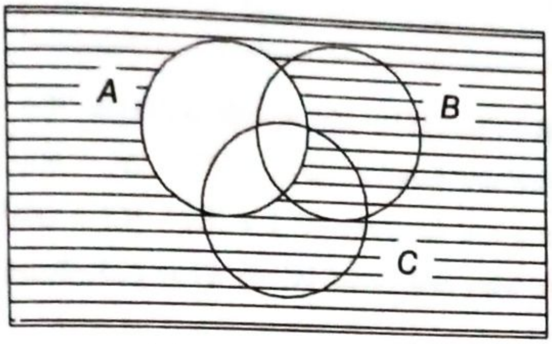
Let the sets A, B, C, and E be given as follows:

E = all students enrolled in the university cricket club.

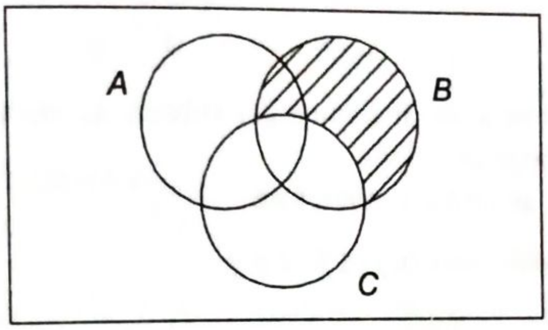
A = male students, B = bowlers, and C = batsmen.

Draw individual Venn diagrams to illustrate (a) female students (b) bowlers who are not batsmen (c) female students who can both bowl and bat.

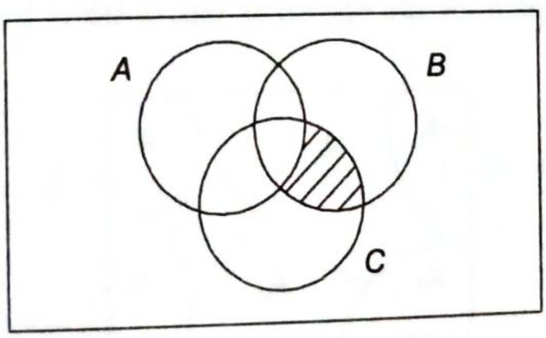
Solution



(a) Female students



(b) Bowlers who are not batsmen



(c) Female students who can both bowl and bat

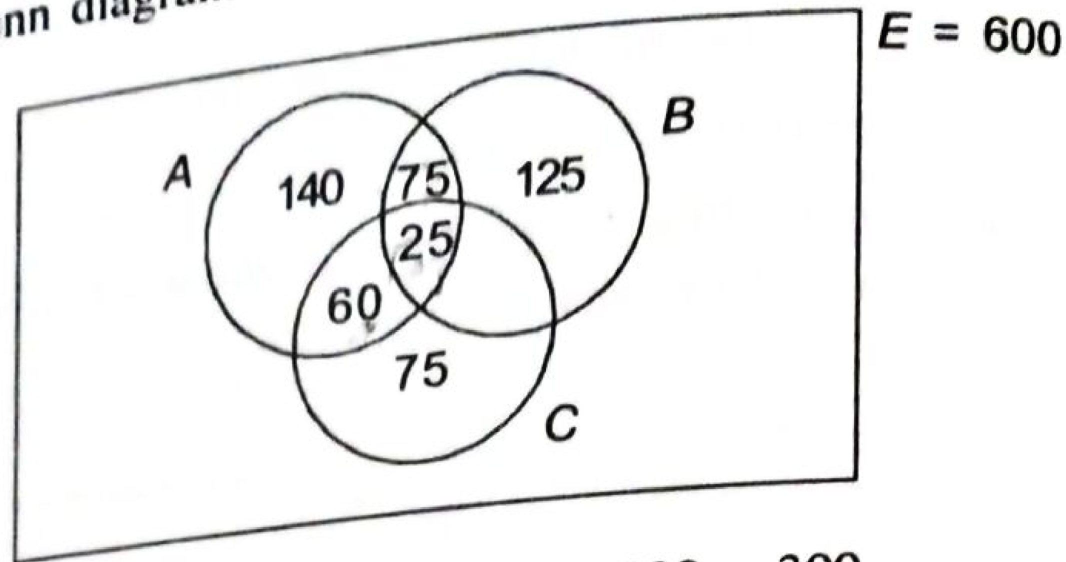
Example 6.3

In Example 6.2, assume that $|E| = 600$, $|A| = 300$, $|B| = 225$, $|C| = 160$. Also, let the number of male students who are bowlers $(A \cap B)$ be 100, 25 of whom are batsmen too $(A \cap B \cap C)$, and the total number of male students who are batsmen $(A \cap C)$ be 85.

Determine the number of students who are: (i) Females, (ii) Not bowlers, (iii) Not batsmen, (iv) Female students who can bowl.

Solution

From the given data, the Venn diagram obtained is as follows:



(i) No. of female students $|A^c| = |E| - |A| = 600 - 300 = 300$

(ii) No. of students who are not bowlers $|B^c| = |E| - |B| = 600 - 225 = 375$

(iii) No. of students who are not batsmen $|C^c| = |E| - |C| = 600 - 160 = 440$

(iv) No. of female students who can bowl $|A^c \cap B| = 125$ (from the Venn diagram)